RELATIVITY AND COSMOLOGY I

Problem Set 4 Fall 2024

1. The Levi-Civita Tensor

In this exercise, we draw the important distinction between the Levi-Civita symbol, which we denoted $\tilde{\epsilon}$, and the Levi-Civita tensor, ϵ .

(a) By using the properties that you proved about the Levi-Civita symbol in Problem Set 2, show that under a coordinate transformation

$$\tilde{\epsilon}_{\mu'_1\cdots\mu'_n} = \det\left(\frac{\partial x'}{\partial x}\right)\frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}}\tilde{\epsilon}_{\mu_1\cdots\mu_n} \,. \tag{1}$$

Objects that transform like this are not tensors, but rather **tensor densities** . Why did we not notice anything problematic in Minkowski spacetime ?

(b) Given how the metric $g_{\mu\nu}$ transforms under a coordinate transformation, show that its determinant transforms as follows

$$g(x') = \left(\det\left(\frac{\partial x'}{\partial x}\right)\right)^{-2} g(x)\,, \tag{2}$$

where we use the standard notation that $g \equiv \det(g_{\mu\nu})$.

(c) Show that

$$\epsilon_{\mu_1\cdots\mu_n} \equiv \sqrt{|g|}\tilde{\epsilon}_{\mu_1\cdots\mu_n} \tag{3}$$

transforms like a tensor for $\det\left(\frac{\partial x'}{\partial x}\right) > 0$. This is the definition of the **Levi-Civita Tensor** .

(d) From the definition (3), show that

$$\epsilon^{\mu_1\cdots\mu_n} = \frac{1}{\sqrt{|g|}}\tilde{\epsilon}^{\mu_1\cdots\mu_n} \tag{4}$$

(e) In the coordinate basis, we can express

$$\epsilon \equiv \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_n}\,. \tag{5}$$

Show that

$$\epsilon = \sqrt{|g|}d^n x\,, \tag{6}$$

where $d^n x$ is the coordinate n-form

$$d^n x \equiv \mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^{n-1}\,. \tag{7}$$

2. Tensors, Manifolds, Coordinates

Answer the following questions

(a) Given a tensor W_μ, prove that $\partial_\mu W_\nu$ is, in a general spacetime, not a tensor.

(b) Prove that, instead, $\partial_{[\mu} W_{\nu]}$ is a tensor.

(c) Given two vector fields expressed in the coordinate basis $V \equiv V^\mu \partial_\mu$ and $W \equiv W^\nu \partial_\nu$, find the explicit expression of the components of their commutator $[V, W]$.

(d) From the definition of the hodge star:

$$(*A)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{n-p}} A_{\nu_1 \cdots \nu_p} \,, \tag{8}$$

argue that it defines a duality between p-forms and $(n-p)$-forms.[1]

3. The Dimensionality of a Space

What is the number of dimensions of the spaces described by these metrics?

$$\begin{aligned} ds^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - (\mathrm{d}x\mathrm{d}y + \mathrm{d}y\mathrm{d}x) + (\mathrm{d}x\mathrm{d}z + \mathrm{d}z\mathrm{d}x) - (\mathrm{d}y\mathrm{d}z + \mathrm{d}z\mathrm{d}y) \,, \\ ds^2 &= \mathrm{d}r^2 + r^2 \Big(\mathrm{d}\theta_1^2 + \mathrm{d}\theta_2^2 + (\mathrm{d}\theta_1 \mathrm{d}\theta_2 + \mathrm{d}\theta_2 \mathrm{d}\theta_1) + \sin^2\theta_1 \cos^2\theta_2 \mathrm{d}\phi^2 \\ &\quad + \sin^2\theta_2 \cos^2\theta_1 \mathrm{d}\phi^2 + \frac{1}{2} \sin 2\theta_1 \sin 2\theta_2 \mathrm{d}\phi^2 \Big) \,. \end{aligned} \tag{9}$$

4. Electromagnetism with p-forms

In the lectures you have seen that Maxwell's equations can be expressed in index-free notation in terms of p-forms. In this exercise, we will work in 4-dimensional Minkowski space.

(a) Check that, in general, $\mathrm{d} * F = *J$ and $\mathrm{d}F = 0$ reproduce Maxwell's equations. **Hint** Remember that exterior derivatives act like

$$(\mathrm{d}A)_{\mu_1 \cdots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2, \cdots \mu_{p+1}]} \,. \tag{10}$$

(b) Show that Maxwell's equations imply $\mathrm{d} * J = 0$. What is this equation ? [2]

In Minkowski space expressed with spherical polar coordinates

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi) \,,$$

suppose $*F = q \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$, where q is a constant.

[1] Hint: show that acting twice with $*$ leads you to the original p-form, and that the dimensionality of the space of p-forms is the same as the dimensionality of the space of $n-p$-forms.

[2] It will be useful to remember that $\epsilon^{\mu\nu\rho\sigma} T_{[\mu\nu\rho]} = \epsilon^{\mu\nu\rho\sigma} T_{\mu\nu\rho}$ for any tensor T and any number of antisymmetrized indices being contracted with the Levi-Civita tensor.

(b) Show that $F = -\frac{q}{r^2}\mathrm{d}t \wedge \mathrm{d}r$,

(c) What are the electric and magnetic fields equal to, for this solution?

(d) Use Stokes' theorem to evaluate $\int_V \mathrm{d} * F$, where V is a ball of radius R in Euclidean three-space. You should retrieve Gauss' law.